

Generalized Hyperfocused Arcs in $PG(2, p)$

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Abstract

A *generalized hyperfocused arc* \mathcal{H} in $PG(2, q)$ is an arc of size k with the property that the $k(k-1)/2$ secants can be blocked by a set of $k-1$ points not belonging to the arc. We show that if q is a prime and \mathcal{H} is a generalized hyperfocused arc of size k , then $k = 1, 2$ or 4 . Interestingly, this problem is also related to the (strong) cylinder conjecture [2], as we point out in the last section.

1 Introduction and preliminaries

Let $PG(2, q)$ be the projective plane over F_q , the finite field with q elements. A k -arc in $PG(2, q)$ is a set of k points with no 3 on a line. A line containing 1 or 2 points of a k -arc is said to be a tangent or secant to the k -arc, respectively.

A *blocking set* of a family of lines \mathcal{F} is a point-set $\mathcal{B} \subset PG(2, q)$ having non-empty intersection with each line in \mathcal{F} . If this is the case, we also say that the lines in \mathcal{F} are *blocked* by \mathcal{B} .

A *generalized hyperfocused arc* \mathcal{H} in $PG(2, q)$ is a k -arc with the property that the $k(k-1)/2$ secants can be blocked by a set \mathcal{B} of $k-1$ points not belonging to the arc. Points of the arc \mathcal{H} will be called *white points* and points of the blocking set \mathcal{B} *black*. In case $k > 1$, since every secant to the arc contains a unique black point, the $k-1$ black points induce a factorization, i.e. a partition into matchings, of the white k -arc and k is forced to be even. For $k = 2$, we only have a trivial example: \mathcal{B} consists of a unique point out of \mathcal{H} on the line through the two points of \mathcal{H} .

An non trivial example of generalized hyperfocused arc is any 4-arc of white points with its three black diagonal points and our main result is that this is the only non trivial example, provided q is an odd prime.

For q even, there are many examples with all black points on a line; in this case \mathcal{H} is simply called a *hyperfocused arc*. As a consequence of the main result of [3], hyperfocused arcs only exist if q is even. Their study is motivated by a relevant application to cryptography in connection with constructions of efficient secret sharing schemes [7, 8]. When q is even, a nice result is that generalized hyperfocused arcs contained in a conic are hyperfocused [1]; moreover it is known that there exist examples of generalized hyperfocused arcs which are not hyperfocused [6]. However, although much more is known about hyperfocused arcs, there are still many open problems concerning them [1, 5, 6].

Let us conclude this section recalling some results on so-called dual 3-nets that will play a crucial role in the proof of our main theorem. A *dual 3-net embedded in $PG(2, \mathbb{F})$* , the projective plane over a field \mathbb{F} , is a triple $\{A, B, C\}$ with A, B, C pairwise disjoint point-sets of size n , called *components*, such that every line meeting two distinct components meets each component in precisely one point.

Example 1. Let ℓ and Γ be a line and a non singular conic in $PG(2, \mathbb{F})$, respectively. It is well known that an abelian group G on $\Gamma \setminus \ell$ can be defined in the following way. Choose a point $O \in \Gamma \setminus \ell$ as the identity of the group and, for any two points $P, Q \in \Gamma \setminus \ell$, let R' be the point that the line through P, Q has in common with ℓ . Then the sum of P and Q is defined by $P + Q = R$, where R is the second of the two points (counted with multiplicity) common to the line OR' and $\Gamma \setminus \ell$. Now, given a proper subgroup A of G of finite order n and one of its cosets B ($\neq A$), the set C of the points of ℓ on some line intersecting both A and B has exactly m points. Then the triple $\{A, B, C\}$ is a dual 3-net of order n embedded in $PG(2, \mathbb{F})$.

The following theorem follows from the main result of [4] (the case that $A \cup B$ is contained in the union of two lines is also characterized there).

Theorem 2. Let $\{A, B, C\}$ be a dual 3-net of order n in $PG(2, \mathbb{F})$. Then, if C is contained in a line and \mathbb{F} has positive characteristic $p \geq n$, $A \cup B$ is contained in a conic. If this conic is irreducible then it is of type described in Example 1.

2 The problem and the main result

In $PG(2, p) = PG(2, \mathbb{F})$, p a prime, let us fix a projective frame, so that every point has homogeneous coordinates $(x : y : z)$. If $A = (a_1, a_2, a_3)$ is a non-zero vector of \mathbb{F}^3 , we denote by $[A] = \langle (a_1, a_2, a_3) \rangle$ the point of $PG(2, p)$ with homogeneous coordinates $(a_1 : a_2 : a_3)$. Sometimes, abusing notation, we will just write A instead of $[A]$ and, in this case, we mean that for the point $[A]$ we are considering the coordinates (a_1, a_2, a_3) or some other special ones, that should be clear from the context. With this notation we write $A \stackrel{\wedge}{=} B$ if $[A] = [B]$, i.e. $A = \lambda B$, for some non zero λ in \mathbb{F} .

Throughout this section $\mathcal{H} = \{W_1, W_2, \dots, W_{2n}\}$ will denote a generalized hyperfocused arc of order $2n$ in $PG(2, p)$, with black point-set \mathcal{B} . W will stand for a projective point, and E , with $W = [E]$ a corresponding suitably chosen representing vector. If W_i, W_j , with $W = [E]$ for some suitable representative E , are two white points, we denote by B_{ij} the unique black point on the line $\langle W_i, W_j \rangle$ and we define b_{ij} by

$$B_{ij} \stackrel{\wedge}{=} E_i + b_{ij}E_j.$$

Obviously, since $B_{ji} = B_{ij}$ we have $b_{ji} = 1/b_{ij}$.

From now on, **we will assume** $p \neq 2, 3$ since our results are trivial in these two cases.

Lemma 3. For $i \neq j \neq k \neq i$ we have

$$b_{ij}b_{jk}b_{ki} = 1 \quad (1)$$

Proof. The proof makes essentially use of the computational technique of Segre's lemma of tangents. For a (cyclicly ordered) white triple $W_i (= [E_i])$, W_j , W_k define the white products: $\omega_i = \prod w_j/w_k$, (and ω_j , ω_k cyclicly) with one factor for every white point $W \triangleq w_i E_i + w_j E_j + w_k E_k$ different from W_i , W_j and W_k (since the white points form an arc w_i, w_j and $w_k \neq 0$). We obviously have $\omega_i \omega_j \omega_k = 1$. Define black products $\beta_i = \prod b_j/b_k$, (and cyclicly) with one factor for every black point $B \triangleq b_i E_i + b_j E_j + b_k E_k$, not on one of the sides of the triangle $W_i W_j W_k$, so this excludes the points B_{jk} , B_{ki} and B_{ij} . We again have $\beta_i \beta_j \beta_k = 1$. Now $\omega_i = \beta_i/b_{jk}$, because for every white point $W \triangleq w_i E_i + w_j E_j + w_k E_k$ there is a (unique) black point $B \triangleq b_i E_i + b_j E_j + b_k E_k$ (on the line $\langle W, W_i \rangle$) with $w_j/w_k = b_j/b_k$, and all these black points contribute to the product defining β_i , except $B_{jk} \triangleq E_j + b_{jk} E_k$ because that is on a side of the triangle. It follows that $\omega_i = \beta_i/b_{jk}$, and with it $b_{ij}b_{jk}b_{ki} = 1$. \square

Now, for $i, j \neq 1$ we have $B_{ij} \triangleq E_i + b_{ij} E_j$ and, applying (1) to the cyclicly ordered triple (E_1, E_i, E_j) , we get

$$B_{ij} \triangleq b_{1i} E_i + b_{1j} E_j,$$

for all $i, j = 2, \dots, 2n$. Taking into account that

$$B_{1j} \triangleq E_1 + b_{1j} E_j \quad \text{and} \quad B_{i1} \triangleq E_i + b_{i1} E_1 \triangleq E_1 + b_{1i} E_i,$$

if we rescale our E_s to $b_{1s} E_s$ (which simply means choosing a more suitable representative for W_s), for each $s \neq 1$, we get

$$B_{ij} \triangleq E_i + E_j, \quad (2)$$

for all $i, j = 1, 2, \dots, 2n$.

Let ℓ be a line intersecting \mathcal{B} in exactly $m < p$ points and set

$$S = \ell \cap \mathcal{B} = \{B_1, B_2, \dots, B_m\}.$$

For a fixed white point $W \in \mathcal{H}$, define the sets

$$S_- = \{W_1^-, W_2^-, \dots, W_m^-\} \quad \text{and} \quad S_+ = \{W = W_1^+, W_2^+, \dots, W_m^+\}$$

where W_i^- is the unique white point other than W on the line $\langle W, B_i \rangle$ and W_i^+ the unique white point other than W_1^- on the line $\langle W_1^-, B_i \rangle$, $i = 1, \dots, m$. In this way, using appropriate white points, \mathcal{H} can be partitioned into pairs of blocks, each block of size m , as we will show below, so m must be a divisor of n .

If $B = B_{ij}$ is the black point on the line $\langle W_i^+, W_j^- \rangle$, $i, j = 1, 2, \dots, m$, then, fixing coordinates of the black points on ℓ so that $B_i = E_1^+ + E_i^- = E + E_i^-$:

$$B = B_{ij} \triangleq E_i^+ + E_j^- \triangleq B_i - E_1^- + B_j - E_1^+ \triangleq \alpha B_i + \beta B_j - (E_1^- + E_1^+) = \alpha B_i + \beta B_j + \gamma B_1,$$

for some constants α, β, γ . So B is on the line ℓ . It follows that $S \cup S^- \cup S^+$ is a dual 3-net of order m and, by Theorem 2, $S^- \cup S^+$ is a set of $2m$ points on a conic Γ and m is the size of a subgroup of a group defined on $\mathcal{H} \setminus \ell$ (cf. Example 1). It follows that m divides $p+1$ or $p-1$ according to the fact that Γ has 0 or 2 points in common with ℓ , respectively. According to this two cases, we say that \mathcal{H} is of *elliptic* or *hyperbolic* type, respectively. Note that the case $|\mathcal{H} \cap \ell| = 1$ cannot occur, otherwise m should divide p .

Our last results can be summarized in the following lemma.

Lemma 4. If a line ℓ intersects \mathcal{B} in exactly $m < p$ points, then m divides n . Moreover, m divides $p-1$ or $p+1$ according to \mathcal{H} is of hyperbolic or elliptic type w.r.t. ℓ , respectively.

Lemma 5. If a line ℓ intersects \mathcal{B} in exactly $m < p$ points, then $m \leq 4$.

Proof. Suppose $m \geq 5$ and consider (homogeneous) projective coordinates $(x : y : z)$ in $PG(2, p)$ in a way that the equation of ℓ is $z = 0$. If \mathcal{H} is of hyperbolic type w.r.t. ℓ , then we can assume that the conic Γ has equation $xy = z$ and, by Theorem 2, we may take

$$S_+ = \{(u : \frac{1}{u} : 1) : u \in \mathbb{F}^* \text{ and } u^m = 1\}.$$

We can also assume that S_+ contains the point $E_0 = (1, 1, 1)$ and the four distinct points of type $E_1 = (v, 1/v, 1)$, $E_{-1} = (1/v, v, 1)$, $E_2 = (v^2, 1/v^2, 1)$, $E_{-2} = (1/v^2, v^2, 1)$, where v is a primitive m -root of unity. Then the two different black points $B_1 \triangleq E_{-1} + E_1 = ((v+1)/v, (v+1)/v, 2)$, $B_2 \triangleq E_{-2} + E_2 = ((v^2+1)/v^2, (v^2+1)/v^2, 2)$ and the white point $W_0 = [E_0]$ are on the line $x = y$, a contradiction.

In case \mathcal{H} is of elliptic type w.r.t. ℓ , we identify in the standard way the affine plane $AG(2, p) = PG(2, p) \setminus \ell$ with the field F_{p^2} . Recall that under this identification three (different) points $a, b, c \in AG(2, p)$ are collinear iff $(a - c)^{p-1} = (b - c)^{p-1}$. If we denote by $(x; 1)$, with $x \in F_{p^2}$, the coordinates of affine points of $PG(2, p)$, we can assume that the conic Γ has equation $x^{p+1} = 1$ and we may take

$$S_+ = \{(x; 1) : x^m = 1\}.$$

Remark that every element of S_+ is forced to be an element of Γ because m divides $p+1$. Now we can assume that S_+ contains the points (corresponding to) $E_0 = (1; 1)$ and four distinct points of type $E_1 = (v; 1)$, $E_{-1} = (1/v; 1)$, $E_2 = (v^2; 1)$, $E_{-2} = (1/v^2; 1)$, where v is a primitive m -th root of unity. Again the claim is that we get a line with two black points and a white one. Actually, the points $W_0 = [E_0]$, $B_1 \triangleq E_{-1} + E_1 = (1/v + v; 2) = (v^p + v; 2)$ and $B_2 \triangleq E_{-2} + E_2 = (1/v^2 + v^2; 2) = (v^{2p} + v^2; 2)$ are on the line whose affine part has equation $x^p = x$. \square

A 4-set $\{i, j, k, l\}$ of indices, or $\{E_i, E_j, E_k, E_l\}$ of points of \mathcal{H} is said to be *special* if $E_i + E_j + E_k + E_l = \mathbf{0}$. In this case, the white points E_i, E_j, E_k, E_l determine exactly three black ones, namely $B_{ij} = B_{kl}$, $B_{jl} = B_{ik}$ and $B_{il} = B_{jk}$, and of course $\{E_i, E_j, E_k, E_l\}$ is a generalized hyperfocused arc contained in \mathcal{H} .

Lemma 6. Let $E_1, E_2, E_3, E_4, E_5, E_6$ be six distinct points of \mathcal{H} such that

$$E_1 + E_2 \stackrel{\wedge}{=} E_3 + E_4 \stackrel{\wedge}{=} E_5 + E_6 \stackrel{\wedge}{=} B.$$

Then, if $\{E_1, E_2, E_3, E_4\}$ and $\{E_1, E_2, E_5, E_6\}$ are special, $\{E_3, E_4, E_5, E_6\}$ is not.

Proof. If $E_1 + E_2 + E_3 + E_4 = E_1 + E_2 + E_5 + E_6 = \mathbf{0}$ then $E_3 + E_4 = E_5 + E_6$ so $E_3 + E_4 + E_5 + E_6 \neq \mathbf{0}$ since $p > 2$ and $E_3 + E_4 \neq \mathbf{0}$. \square

We are now in the right position in order to prove our main result.

Theorem 7. If p is an odd prime and \mathcal{H} is a hyperfocused arc of size $2n$, in $PG(2, p)$ then $n \leq 2$.

Proof. Since $n > 2$, by Lemma 6, we can consider four points E_i with $E_1 + E_2 \stackrel{\wedge}{=} E_3 + E_4$ but with $E_1 + E_2 + E_3 + E_4 \neq \mathbf{0}$.

Then we have 5 different black points: $B_{12} = B_{34}$, B_{13} , B_{14} , B_{23} and B_{24} , moreover $B_{12} = B_{34}$, B_{13} , and B_{24} are collinear, and also $B_{12} = B_{34}$, B_{14} and B_{23} are collinear, and these two lines are different (for otherwise we would have a line with more than 4 black points). We first show that $B_{13} \neq B_{24}$. If they were equal, then we would have $B_{13} \stackrel{\wedge}{=} E_1 + E_3 \stackrel{\wedge}{=} E_2 + E_4 \stackrel{\wedge}{=} E_1 + E_2 + E_3 + E_4 \stackrel{\wedge}{=} B_{12}$. The same argument shows that $B_{14} \neq B_{23}$. Next we show that $B_{12} = B_{34}$, B_{13} and B_{24} are collinear. This is clear since the vector $E_1 + E_2 + E_3 + E_4$ (representing $B_{12} = B_{34}$) is a linear combination of $E_1 + E_3$ and $E_2 + E_4$.

After a suitable linear transformation we may put $B_{12} = B_{34} = (0, 0, 1)$, $B_{13} = (1, 0, 0)$, $B_{24} = (1, 0, 1)$, $B_{14} = (0, 1, 0)$ and $B_{23} = (0, 1, 1)$. Remark that since the lines $x = 0$, $y = 0$ and $z = 0$ contain (at least) two black points the coordinates of white points are now all nonzero. If we put $E_1 = (a, b, c)$ then we can compute 'everything'. Indeed, from B_{12} we have $E_2 = (-a, -b, *)$ and from B_{14} we have $E_4 = (-a, *, -c)$. Now by B_{24} we get $E_2 = (-a, -b, c - 2a)$ and $E_4 = (-a, b, -c)$. From B_{13} we find $E_3 = (*, -b, -c)$ and from $B_{34} = B_{12}$ we find $E_3 = (a, -b, -c)$. Finally, from B_{23} we have $a = b$, and hence $E_1 = (a, a, c)$, $E_2 = (-a, -a, c - 2a)$, $E_3 = (a, -a, -c)$ and $E_4 = (-a, a, -c)$. Note that our configuration has an $x \leftrightarrow y$ symmetry. We proceed to show that in fact one of the two lines $x = 0$ or $y = 0$ is a 3-secant, and the other one is a 4-secant.

Let start by looking at the case that both lines are a 3-secant. We then find additional white points E_5 and E_6 on one conic and E'_5 and E'_6 on the other: $(a - 2c, a, 2a - c)$ and $(2c - a, -a, c)$ on the conic with respect to the line $y = 0$ as well as $(a, a - 2c, 2a - c)$ and $(-a, 2c - a, c)$, on the conic with respect to $x = 0$. Adding two white points produces a black point, and we find black points $(2a - 2c, 2a - 2c, 4a - 2c)$ and $(2c - 2a, 2c - 2a, 2c)$. Since the only black point on the line $x = y$ is $(0, 0, 1)$, then $a = c$, i.e. $E_1 = E_2$, a contradiction.

Next we assume that the line $x = 0$ is a 4-secant and let B be the fourth black point on it. We label the black points on this line with $B_{12} = B_{34} = (0, 0, 1) = \infty$, $B_{14} = (0, 1, 0) = 0$, $B_{23} = (0, 1, 1) = 1$, and $B = (0, 1, b) = b$, with $b \neq 0, 1, \infty$. Let W_5, \dots, W_8 be the remaining white points on the conic determined by the line $x = 0$ and the point W_1 , with

W_1, W_3, W_5, W_7 forming one component, and the remaining W_i the other. For a suitable ordering of the W_i we then find the following table for (W_i, W_j, B_{ij}) .

	W_1	W_3	W_5	W_7
W_2	∞	1	0	b
W_4	0	∞	b	1
W_6	1	b	∞	0
W_8	b	0	1	∞

From $W_2 + W_5 \stackrel{\wedge}{=} 0 = (0, 1, 0)$ we get $W_5 = (a, u, 2a - c)$.

From $W_8 + W_3 \stackrel{\wedge}{=} 0 = (0, 1, 0)$ we get $W_8 = (-a, v, c)$.

From $W_5 + W_8 \stackrel{\wedge}{=} 1 = (0, 1, 1)$ we get $v = 2a - u$.

From $W_5 + W_4 \stackrel{\wedge}{=} b = (0, 1, b)$ we get $(1, b) \stackrel{\wedge}{=} (a + u, 2a - 2c)$.

From $W_8 + W_1 \stackrel{\wedge}{=} b = (0, 1, b)$ we get $(1, b) \stackrel{\wedge}{=} (3a - u, 2c)$.

Since $a \neq 0$ we can compute $u = 3a - 4c$.

Finally $b = (0, 1, b) \stackrel{\wedge}{=} W_4 + W_5 = (0, a + u, 2a - 2c) = (0, 4a - 4c, 2a - 2c)$, so $b = 1/2$.

Next we consider the case that both lines ($y = 0$ and $x = 0$) are a 4-secant: because of x, y symmetry we then also find points W'_j , $j = 5, \dots, 8$, in particular we find $W'_5 = (3a - 4c, a, 2a - c)$ and hence a black point $B_{55'} \stackrel{\wedge}{=} (4a - 4c, 4a - 4c, 4a - 2c)$ on the line $x = y$, but the only black point on this line is $(0, 0, 1)$, hence $a = c$ and the whole thing collapses.

So we find that one line is a 4-secant, and the other a 3-secant.

So summarizing: if $E_1 + E_2 \stackrel{\wedge}{=} E_3 + E_4$ but their sum is nonzero, then the points $B_{12} = B_{34}, B_{14}, B_{23}$ are collinear and so are the points $B_{12} = B_{34}, B_{13}, B_{24}$, one of these lines is a three-secant, and the other a four secant, with fourth point B , determining $1/2$, that is if $B_{12} = B_{34} = \infty$, $B_{14} = 0$, $B_{23} = 1$, is the four-secant, then $B = 1/2$. Note that the role of $B_{12} = B_{34}$ is special, but B_{23} and B_{14} can be interchanged, indeed, the (Möbius)-transformation fixing ∞ and interchanging 1 and 0 is $x \mapsto 1 - x$ and this fixes $1/2$, as it should.

We have seen that a non-special fourtuple $\{W_i, W_j, W_k, W_l\}$ with $B \stackrel{\wedge}{=} E_i + E_j \stackrel{\wedge}{=} E_k + E_l$ produces a 3-secant ℓ and a 4-secant m intersecting in B , but more important: we can find back the pairs (W_i, W_j) and (W_k, W_l) from the triple (ℓ, m, B) . The four black points on the 4-secant m fall into three groups: the intersection point B , a pair C_1, C_2 , and the fourth point D , the midpoint of the other two when we put the intersection point at infinity, or in other words the cross ratio $(B, C_1; C_2, D) = (B, C_2; C_1, D) = 1/2$. Fixing coordinates as we did we found that W_1 and W_2 are on the line $x = y$ and W_3 and W_4 are on the line $x = -y$, and these lines of course contain no further white points.

Now start with a fixed 3-secant on a fixed black point B . The 3-secant determines a partition of the n lines connecting B with the white points in $n/3$ triples. Every triple determines 3 pairs of pairs (W_i, W_j) and (W_k, W_l) and the fourtuple (W_i, W_j, W_k, W_l) is not special, because the sum of the six white point $W_i, W_j, W_k, W_l, W_m, W_n$ on the conic

determined by the 3-secant and any of the W 's add up to zero (that is, the corresponding E 's do).

So this pair of pairs determines a configuration where one of the lines is the fixed 3-secant, and the other a 4-secant. Now, every 4-secant (intersecting our fixed 3-secant in the fixed black point B) gives (at most) 1 configuration, as we saw above, so we find at least $(n/3) \times 3 = n$ different 4-secants through B , containing together (apart from B) at least $3n$ black points, contradiction.

So all four tuples are special, but this means $2n \leq 4$. \square

3 Generalizing further

Generalized hyperfocused arcs were introduced as a generalization of hyperfocused arcs, but this is not how we hit upon them. In fact we see them as a special case of a configuration whose study is motivated by the (strong) cylinder conjecture [2]. This conjecture states that a set C of q^2 points in $AG(3, q)$, the affine 3-space over F_q , that intersects every plane in $0 \bmod q$ points must be a cylinder, the union of q parallel lines. The ordinary conjecture states this for q is prime, the strong version for prime powers.

In an attempt to prove this conjecture we were led to the study of configurations in $PG(2, p)$ consisting of a set \mathcal{W} of $k \leq (p+1)/2$ 'white' points, and a *multiset* \mathcal{B} of $k-1$ 'black' points with the property that every line containing $m > 0$ white points contains exactly $m-1$ black points (counted with multiplicity). The connection is as follows.

Embed $AG(3, q)$ in $PG(3, q)$ and let P be a point of $AG(3, q)$ not in C . Project C on to a plane $PG(2, q)$ not containing P . We get a multiset of q^2 points in $PG(2, q)$ with a multiple of q points on every line.

Let the *weight* of a line ℓ in $PG(2, q)$ be $k-1$, if the plane $\langle P, \ell \rangle$ has kq points. Consider a line of weight -1 . Clearly all points of it have weight 0, that is, they don't occur in the multiset, or equivalently, they occur in the multiset with multiplicity zero.

If we add up the weights of the lines through a zero-point, we get -1 . So points on the intersection of two (-1) -lines must be on a weight > 0 line as well. More precisely, if Q is a point on k (-1) -lines, then the weights of the positive lines through Q add up to $k-1$.

Now, dualizing, we get two disjoint point-sets: a set of *white points*, corresponding to the duals of the (-1) -lines, and a multi-set of *black points* corresponding to the duals of the positive weight lines, the multiplicity of a point being the weight of its corresponding line. Every line that contains more than one white point contains the appropriate number of black points. Every line having exactly one white point has no black points. The total number of black points is one less than the number of white points. In the example of the cylinder all black and white points are collinear; so our problem is to find (and exclude) the other configurations. Remark that we may choose P in such a way that the number of lines with weight different from 0 is at most q (there are at most $q^2 - q$ planes that intersect C in a number of points different from q , choose P such that it is on relatively few of them).

In case the set \mathcal{W} is an arc, the multiset \mathcal{B} is an ordinary set and we are looking at a generalized hyperfocused arc. Our main result therefore is that this situation essentially does not occur. To classify these configurations in general seems to be hopeless but the prime case could be doable (and might settle the cylinder conjecture!). Here we give the relevant (that is, with at most $(p+1)/2$ white points) examples we know in planes of prime order. In general many more examples exist, also in planes of prime order.

Example 8. (White and black points are collinear) All white points are collinear, black points are arbitrary other points on this line, the right number of them. This is the only example coming from cylinders.

Example 9. (White points on two lines) In $AG(2, p)$ consider as white points $(a, 0)$ and $(0, b)$ where a and b are in a (multiplicative) subgroup of $GF(p)^*$ of order n say (or in a coset). Take black points at infinity in the points $(a : -b : 0) = (1 : -b/a : 0)$, and take the origin with multiplicity $n - 1$.

This example is characterized by the property that the white points are contained in the union of two lines, for example by using the earlier mentioned result on 3-nets [4] (but in its general form).

Example 10. (White points form an arc) The white points form a 4-arc, and there are 3 black points, the diagonal points. This is the example classified in this note.

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